Solutions to Question Sheet 5, Continuity II v1 2019-20

Intermediate Value Theorem continued

1. Assume that $f:[0,1]\rightarrow[0,1]$ is continuous on $[0,1]$. Prove that there exists some $c\in[0,1]$ such that
$$f(c)=\sin\left(\frac{\pi c}{2}\right).$$
Hint Apply the Intermediate Value Theorem to
$$h(x)=f(x)-\sin\left(\frac{\pi x}{2}\right).$$
Solution Follow the hint and let
$$h(x)=f(x)-\sin\left(\frac{\pi x}{2}\right).$$
We are told that $f:[0,1]\rightarrow[0,1]$ which means that $0\leq f(x)\leq 1$ for all $x\in[0,1]$. Thus
$$h(0)=f(0)-0\geq 0 \quad \text{while} \quad h(1)=f(1)-1\leq 1-1=0.$$
That is $h(0)\geq 0\geq h(1)$. If $h(0)=0$ choose $c=0$, while if $h(1)=0$ choose $c=1$. Otherwise apply the Intermediate Value Theorem to h with $\gamma=0$ to find $c\in(0,1)$ for which $h(c)=0$.

2. (Generalising Question 1.) Prove a version of the *Fixed Point Theorem*. If $f,g:[a,b]\rightarrow[a,b]$ are continuous functions such that $f(a)\geq g(a)$ and $f(b)\leq g(b)$ then there exists $c\in[a,b]$ such that $f(c)=g(c)$.

   Hint: Follow the hint in the previous question and consider $h(x)=f(x)-g(x)$.

   Solution Let $h(x)=f(x)-g(x)$, a continuous function on $[a,b]$ by the sum rule for continuous functions.

   The assumption $f(a)\geq g(a)$ implies $h(a)=f(a)-g(a)\geq 0$. If $h(a)=0$ then $f(a)=g(a)$ and we are finished. So assume $h(a)>0$.

   Also the assumption $f(b)\leq g(b)$ implies $h(b)=f(b)-g(b)\leq 0$. Again if $h(b)=0$ then $f(b)=g(b)$ and we are finished. So assume $h(b)<0$.

   Thus, we are left assuming $h(a)>0>h(b)$ when we can apply the Intermediate Value Theorem to h on $[a,b]$ with $\gamma=0$ to find $c\in(a,b)$ for which $h(c)=0$, i.e. $f(c)=g(c)$.

Boundedness Theorem

3. Recall the **Boundedness Theorem** which states that a *continuous function on a closed bounded interval is bounded and attains its bounds.* In this question we check if the conditions that the function be *continuous* on a *closed, bounded interval* are necessary. So, if remove any of these conditions does the conclusion of the Theorem still hold?

   Give examples of

   i) A function on a closed bounded interval that is not bounded.

   ii) A continuous function on $(-1,1)$ with range $(-\infty,\infty)$, (and thus is not bounded).

   iii) A function on $[0,1]$ that is bounded but does not attain its bounds.

   Solution i) For an example of a function on a closed interval that is not bounded,

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

   Note that this function is not continuous on $[0,1]$. So we can deduce that the conclusion of the Boundedness Theorem does not necessarily follow if we do not demand f to be continuous. (Make sure you understand all the negations in this last sentence.)

   ii) For an example of a continuous function on $(-1,1)$ with range $(-\infty,\infty)$ we can base our answer on functions unbounded near $x=1$ and -1, e.g. $1/(1-x)$ and $1/(1+x)$ for this makes it possible for the range to be $(-\infty,\infty)$.

   Yet for $-1 < x < 1$ both $1/(1-x)$ and $1/(1+x)$ are positive. So, to get negative values, multiply one of these functions by -1 and then add, i.e.

$$x \longmapsto \frac{1}{1-x} - \frac{1}{1+x} = \frac{2x}{1-x^2}.$$

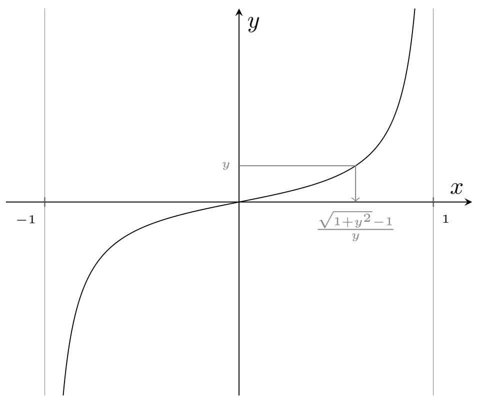
   To see that this has image $(-\infty,\infty)$ you have to be able to solve

$$\frac{2x}{1-x^2} = y$$

for *any* $y \in (-\infty, \infty)$. But rearranging, solving the quadratic and taking the correct root(!) gives the solution

$$x = \frac{\sqrt{1+y^2}-1}{y}.$$

Figure for Question 3ii:



Note that the interval $(-1, 1)$ on which the function is defined is not closed. So we can deduce that the conclusion of the Boundedness Theorem does not necessarily follow if we do not demand that the interval on which the function is defined is closed.

iii) For an example of a function on $[0, 1]$ that does not attain its bounds,

$$h(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1/2 & \text{if } x = 1 \text{ or } x = 0. \end{cases}.$$

Note that this function is not continuous on $[0, 1]$ so again the conclusion of the Boundedness Theorem does not necessarily follow if we do not demand f to be continuous.

To sum up

f continuous on a closed and bounded interval $\implies$ f is bounded,

f continuous on a closed interval $\not\Rightarrow$ f is bounded,

f continuous on a bounded interval $\not\Rightarrow$ f is bounded,

f defined on a closed and bounded interval $\not\Rightarrow$ f is bounded,

4. i) a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x^2 + 1}$$

is bounded for all $x \in \mathbb{R}$.

b) Does f attain its bounds?

c) Is this a counter-example to the Boundedness Theorem, in particular that functions continuous on a closed bounded interval attain their bounds?

ii) a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{x^2 + 1}$$

is bounded for all $x \in \mathbb{R}$.

b) Does f attain its bounds?

iii) Sketch the graphs of both functions.

Hint: Expand and rearrange the inequalities

$$(x-1)^2 \geq 0 \quad \text{and} \quad (x+1)^2 \geq 0.$$

Solution i) a) The function is bounded above because

$$x^2 \geq 0 \implies x^2 + 1 \geq 1 \implies \frac{1}{x^2+1} \leq 1.$$

The function is bounded below because

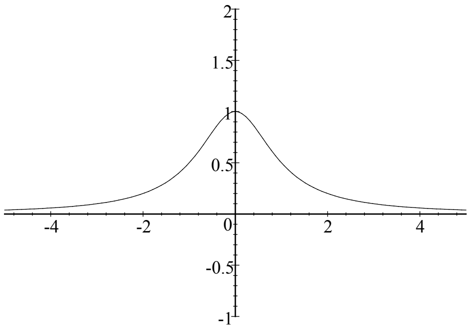
$$x^2 + 1 \text{ is positive} \implies \frac{1}{x^2+1} \text{ is positive, i.e. } \frac{1}{x^2+1} \geq 0.$$

b) This upper bound is attained at $x = 0$. The lower bound is not attained though

$$\lim_{x \to \infty} \frac{1}{x^2+1} = 0,$$

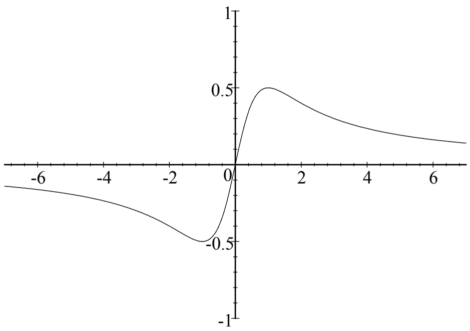
so the function gets arbitrarily close to 0 but it is never attained.

Figure for Question 4i,



c) The function is not a counter-example to the Boundedness Theorem which says that if f is continuous on a closed & bounded interval then it attains it's bounds. In this case the interval on which f is defined is **not** a bounded interval, it is all of $\mathbb{R}$.

ii) a) I suggest start by sketching the graph:



It would appear from the graph that

$$-\frac{1}{2} \leq \frac{x}{x^2+1} \leq \frac{1}{2} \tag{1}$$

for all x but can we prove this?

We prove each inequality separately. For the upper bound

$$\frac{x}{x^2+1} \leq \frac{1}{2} \iff x^2 - 2x + 1 \geq 0 \iff (x-1)^2 \geq 0,$$

which is true. Thus the inequality holds.

The the lower bound in (**??**) ,

$$-\frac{1}{2} \leq \frac{x}{x^2+1} \iff x^2 + 2x + 1 \geq 0 \iff (x+1)^2 \geq 0,$$

which is true. Thus the inequality holds.

Hence $x/(x^2+1)$ is bounded above by $1/2$ and below by $-1/2$.

b) The upper bound is attained at $x = 1$, the lower at $x = -1$

Strictly Monotonic functions

5. Prove that

   i) for all $n \in \mathbb{N}$, with n even, then x^n is strictly increasing on $[0, \infty)$,

   ii) for all $n \in \mathbb{N}$, with n even, then x^n is **not** strictly increasing on $\mathbb{R}$,

   iii) for all $n \in \mathbb{N}$, with n odd, then x^n is strictly increasing on $\mathbb{R}$.

   Hint: use the factorization

   $$x^n - y^n = (x - y)\left(x^{n-1} + yx^{n-2} + y^2x^{n-3} + ... + y^{n-2}x + y^{n-1}\right).$$

   In (iii) it might help to look at 3 cases, $x > y \geq 0$, $x > 0 > y$ and $0 \geq x > y$.

Solution i) Take any $x > y \geq 0$. Then

$$x^n - y^n = (x - y)\left(x^{n-1} + yx^{n-2} + y^2x^{n-3} + \cdots + y^{n-1}\right). \qquad (2)$$

Both terms of the right are positive (which also means non-zero), the first since $x > y$, and the second since all terms are ≥ 0 while $x^n > 0$ implies that it is non-zero. Hence $x^n > y^n$. Thus x^n is *strictly* increasing on $[0, \infty)$.

ii) If n is even then choose, as an example $x = 1$ and $y = -2$ so $x > y$. But $(-2)^n = 2^n > 1^n$ so $x^n \not> y^n$. Hence x^n is **not** increasing on $\mathbb{R}$.

iii) Given $x > y$ there are three possibilities.

- The case $x > y \geq 0$ was dealt with in part (i).
- Assume $x > 0 > y$. The positive power of a positive number is positive, so $x > 0 \implies x^n > 0$. The *odd* power of a negative number is negative, so $0 > y \implies 0 > y^n$. Thus $x > 0 > y \implies x^n > 0 > y^n$, i.e. $x^n > y^n$.
- If $0 > x > y$ then

$$\begin{aligned} 0 > x > y &\implies (-y) > (-x) > 0 \\ &\implies (-y)^n > (-x)^n > 0 \quad \text{by part (i)} \\ &\implies -y^n > -x^n > 0 \quad \text{since } n \text{ is odd,} \\ &\implies x^n > y^n. \end{aligned}$$

Hence in all three cases $x > y \implies x^n > y^n$.

Note That the results of this question were used in the lecture notes as examples of function to which the Inverse Function Theorem applies with the conclusion that $x^{1/n}$ is a strictly increasing continuous function for $x > 0$ for all n and $x \in \mathbb{R}$ for odd n.

6. Prove that the hyperbolic functions $\sinh x$ and $\tanh x$ are strictly increasing on $\mathbb{R}$ while $\cosh x$ is strictly increasing on $[0, \infty)$.

Hint. Prove that

$$\sinh(x + y) > \sinh x$$

for all $x \in \mathbb{R}$ and $y > 0$, similarly for tanh, while

$$\cosh(x + y) > \cosh x$$

for all $x, y > 0$.

Solution From the definition of sinh,

$$\sinh(x + y) > \sinh x \iff e^{x+y} - e^{-x-y} > e^x - e^{-x}.$$

Multiply through by $e^{x+y} > 0$ to get a sequence of equivalences

$$\begin{aligned} e^{x+y} - e^{-x-y} > e^x - e^{-x} &\iff e^{2x+2y} - 1 > e^{2x+y} - e^y \\ &\iff e^{2x+2y} - e^{2x+y} + e^y - 1 > 0 \\ &\iff e^{2x+y}(e^y - 1) + e^y - 1 > 0 \\ &\iff \left(e^{2x+y} + 1\right)(e^y - 1) > 0. \end{aligned}$$

Yet this is a true statement since y *strictly* greater than 0 implies that e^y is *strictly* greater than 1. i.e. $e^y - 1 > 0$, while $e^{2x+y} + 1 > 1 > 0$. Hence sinh is *strictly* increasing on $\mathbb{R}$.

From the definition of tanh we have a series of equivalences,

$$\begin{aligned}
\tanh(x+y) > \tanh x &\iff \frac{e^{x+y} - e^{-x-y}}{e^{x+y} + e^{-x-y}} > \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&\iff \left(e^x + e^{-x}\right)\left(e^{x+y} - e^{-x-y}\right) > \left(e^{x+y} + e^{-x-y}\right)\left(e^x - e^{-x}\right) \\
&\iff e^y - e^{-y} > e^{-y} - e^y \\
&\iff e^{2y} > 1.
\end{aligned}$$

This is true since y *strictly* greater than 0 implies e^{2y} is *strictly* greater than 1. Hence tanh is *strictly* increasing on $\mathbb{R}$.

From the definition of cosh we have the equivalence,

$$\cosh(x+y) > \cosh x \iff e^{x+y} + e^{-x-y} > e^x + e^{-x}.$$

Multiply through by $e^{x+y} > 0$ and rearrange to get the sequence of equivalences

$$\begin{aligned}
e^{x+y} + e^{-x-y} > e^x + e^{-x} &\iff e^{2x+2y} + 1 > e^{2x+y} + e^y \\
&\iff e^{2x+2y} - e^y > e^{2x+y} - 1 \\
&\iff e^y\left(e^{2x+y} - 1\right) > e^{2x+y} - 1 \\
&\iff \left(e^y - 1\right)\left(e^{2x+y} - 1\right) > 0.
\end{aligned}$$

This is true since $y > 0$ and $2x+y > 0$ imply $e^y - 1 > 0$ and $e^{2x+y} - 1 > 0$ and so their product is strictly greater than 0. Hence cosh is *strictly* increasing on $[0, \infty)$.

Inverse Function Theorem

7. State the Inverse Function Theorem.

   Explain how to define the following inverse functions,

   i) $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$,

   ii) $\cosh^{-1} : [1, \infty) \to [0, \infty)$.

   iii) $\tanh^{-1} : (-1, 1) \to \mathbb{R}$.

   Don't forget to show that the inverses map between the sets shown.

   A problem might be that the Inverse Function Theorem as stated in lectures refers to bounded interval while here we have $\mathbb{R}$, $[1, \infty)$ and $[0, \infty)$. An approach might be to take a large N and consider sinh and tanh on $[-N, N]$ and cosh on $[0, N]$, define their inverses and finish by letting $N \to \infty$.

   Solution ***Inverse Function Theorem*** *Assume that f is continuous and strictly monotonic on the closed and bounded interval $[a, b]$. Write*

$$[c, d] = \begin{cases} [f(a), f(b)] & \textit{if } f \textit{ is increasing} \\ [f(b), f(a)] & \textit{if } f \textit{ is decreasing.} \end{cases}$$

   Then there exists a function g, continuous and strictly monotonic on $[c, d]$ which is inverse to f, i.e. $g(f(x)) = x$ for all $x \in [a, b]$ and $f(g(y)) = y$ for all $y \in [c, d]$.

   From Question 4 the functions sinh and tanh are strictly monotonic, in fact increasing, on $\mathbb{R}$ while cosh is strictly monotonic, in fact increasing, on $[0, \infty)$. But neither $\mathbb{R}$ nor $[0, \infty)$ are a closed and bounded interval $[a, b]$ as seen in the Inverse Function Theorem stated here. So, as suggested in the question, let $N \geq 1$ be given.

   The Inverse Function Theorem can be applied to sinh and tanh on $[-N, N]$ and cosh on $[0, N]$ to find the inverses

$$\begin{aligned} \sinh^{-1} &: [\sinh(-N), \sinh N] \to [-N, N], \\ \tanh^{-1} &: [\tanh(-N), \tanh N] \to [-N, N], \\ \cosh^{-1} &: [\cosh 0, \cosh N] \to [0, N]. \end{aligned}$$

Let $N \to +\infty$ noting that

$\sinh N \to +\infty$, $\sinh(-N) = -\sinh N \to -\infty$;

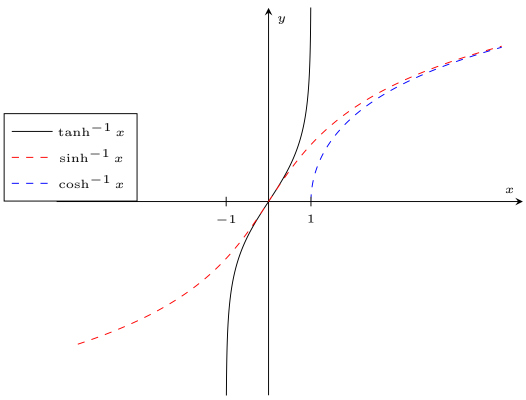
$\tanh N \to 1$, $\tanh(-N) = -\tanh N \to -1$;

$\cosh 0 = 1$ and $\cosh N \to +\infty$.

Thus we get inverses

$$\begin{aligned}\sinh^{-1} &: (-\infty, \infty) \to (-\infty, \infty),\\ \tanh^{-1} &: (-1, 1) \to (-\infty, \infty),\\ \cosh^{-1} &: [1, \infty) \to [0, \infty).\end{aligned}$$

Perhaps from this diagram you can see that the inverse functions map between the sets claimed above.



Continuing, in the question I suggested that you check directly the domains and codomains of these inverses.

i) For $x \in \mathbb{R}$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Let $y \in \mathbb{R}$ be given. Then $\sinh x = y$ is equivalent to

$$\frac{e^x - e^{-x}}{2} = y \iff (e^x)^2 - 2ye^x - 1 = 0.$$

This quadratic has a solution

$$e^x = y + \sqrt{y^2 + 1},$$

the positive root having been taken to ensure that x is real. We can thus find x in terms of y and hence sinh maps $\mathbb{R}$ **onto** $\mathbb{R}$, i.e. $\sinh : \mathbb{R} \to \mathbb{R}$ and $\sinh^{-1}$ maps back from $\mathbb{R}$ to $\mathbb{R}$.

ii) For $\cosh x$ we have

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

It can be checked that for $t \in \mathbb{R}$ we have $t + t^{-1} \geq 2$, with equality when $t = 1$. Thus, with e^x replacing t, we have

$$\cosh x = \frac{e^x + e^{-x}}{2} \geq 1$$

for all $x \in \mathbb{R}$. But can cosh take every value ≥ 1?

Answer Yes, Let $y > 1$ be given. As for sinh the equation $\cosh x = y$ has solutions $e^x = y \pm \sqrt{y^2 - 1}$.

But of these two solutions which should be taken? In the previous question we only showed that $\cosh x$ is strictly increasing for $x \geq 0$. Thus, to be able to apply the Inverse Function Theorem, we must restrict ourselves to $x \geq 0$. Yet $x \geq 0$ implies $e^x \geq 1$. If we chose the negative sign: $e^x = y - \sqrt{y^2 - 1}$, then

$$e^x = \left(y - \sqrt{y^2 - 1}\right) \frac{y + \sqrt{y^2 - 1}}{y + \sqrt{y^2 - 1}} = \frac{1}{y + \sqrt{y^2 - 1}} < 1,$$

since $y > 1$. Thus we would then have both $e^x \geq 1$ and $e^x < 1$, a contradiction.

Therefore we must take the positive sign: $e^x = y + \sqrt{y^2 - 1}$ in which case $x = \ln\left(y + \sqrt{y^2 - 1}\right)$ is **the** solution of $\cosh x = y$. Thus cosh maps $[0, \infty)$ **onto** $[1, \infty)$ and hence $\cosh^{-1} : [1, \infty) \to [0, \infty)$.

iii) Let $y \in \mathbb{R}$ be given. Then $\tanh x = y$ is equivalent to

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y.$$

This rearranges to

$$e^{2x} = \frac{1+y}{1-y}. \tag{3}$$

We can take the logarithm of both sides, and thus find x, as long as $(1+y)/(1-y)$ is positive. In turn this means that we must have either both $1+y$ and $1-y$ positive or both negative.

The second possibility, $1+y<0$ and $1-y<0$ combine as $1<y<-1$ for which there are no y. The first possibility, $1+y>0$ and $1-y>0$ combine as $-1<y<1$. Hence, for such y we can solve for x. Thus tanh maps $\mathbb{R}$ onto $(-1,1)$ and hence $\tanh^{-1} : (-1,1) \to \mathbb{R}$.

Logarithm

8. Prove that the natural logarithm, defined as the inverse of the exponential function, satisfies

$$\ln a + \ln b = \ln ab$$

for all $a, b > 0$.

(As throughout this course you may assume that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$.)

Hint What are $e^{\ln a + \ln b}$ and $e^{\ln ab}$? You may need to use the fact that e^x is an injective function.

Solution From the definition of $\ln a$ as the inverse of e^x we have $ab = e^{\ln(ab)}$, $a = e^{\ln a}$ and $b = e^{\ln b}$. Thus

$$e^{\ln(ab)} = ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b},$$

by the assumption running throughout the course that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$. Yet e^x is a strictly increasing function and thus one-to-one. Recalling that one-to-one means that if $f(x) = f(y)$ then $x = y$ we deduce here that the two powers of e are the same, i.e. $\ln ab = \ln a + \ln b$.

Additional Questions for practice

9. Show that

$$2\cos^2 x + 3\cos x + 1 = 2x^2 + 3x + 1$$

has **a** solution in $[0, \pi/2]$.

Solution Let

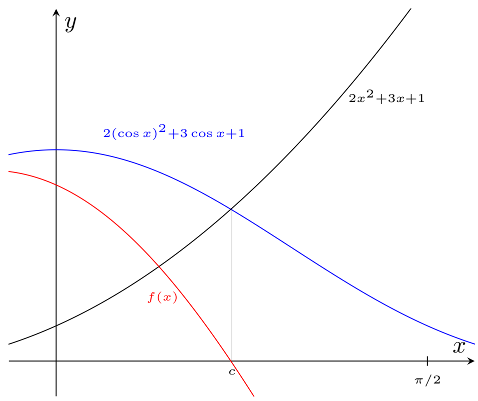
$$\begin{aligned} f(x) &= \left(2\cos^2 x + 3\cos x + 1\right) - \left(2x^2 + 3x + 1\right) \\ &= 2\left(\cos^2 x - x^2\right) + 3\left(\cos x - x\right). \end{aligned}$$

It suffices to find a solution to $f(x) = 0$. Looking at the values of the function at the end points of the interval,

$$f(0) = 5 > 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{2} - 3\frac{\pi}{2} < 0.$$

So, by the Intermediate Value Theorem, there exists $c \in (0, \pi/2)$ for which $f(c) = 0$.

Figure for Question 9,



10. Show that

$$\frac{x}{\sin x} + \frac{1}{\cos x} = \pi$$

has **a** solution with $x \in (0, \pi/2)$.

Solution Again follow the principle of ridding ourselves of fractions by *multiplying up* and solving

$$x\cos x + \sin x = \pi \sin x \cos x.$$

Let

$$f(x) = x\cos x + \sin x - \pi \sin x \cos x.$$

Then $f(0) = 0$. But for $x = 0$ the term $\sin x$ is zero so we cannot divide through by $\sin x$ to get a solution of the original problem.

Instead, let us look at an x near 0 for which $\sin x$ and $\cos x$ are known, for example $\pi/4$. We know that $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$. Thus

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{\pi}{2} = \frac{4 - \pi\left(2\sqrt{2} - 1\right)}{4\sqrt{2}} < 0.$$

Combined with $f(\pi/2) = \pi/2 > 0$ we have $f(\pi/4) < 0 < f(\pi/2)$. Then by the Intermediate Value Theorem with $\gamma = 0$, there exists $c \in (\pi/4, \pi/2)$ for which $f(c) = 0$ For such c the terms $\sin c$ and $\cos c$ are non-zero and we can divide by them to see that c satisfies

$$\frac{c}{\sin c} + \frac{1}{\cos c} = \pi.$$

Figure for Question 10,